Chapter 2. Transversality and Intersection.

§1. Manifolds with Boundary

Defn. A k-dim mfd w/ bdry is a subset $X \subseteq \mathbb{R}^N$ that is locally diffeomorphic to an open subset of $\mathbb{H}^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$.

The boundary ∂X of X consists of pts that corresponds to a pt on $\partial \mathbb{H}^k = \mathbb{R}^{k-1}$ under some (and hence every) parametrization. Interior $\operatorname{Int} X = X \setminus \partial X$.

· $\operatorname{Int} X$ is open in X. ∂X is a closed $(n-1)$-dim submfd of X.

Defn. Tangent space T_xX well-def'd on mfd X w/ bdry. For $f: X \to Y$, $df_x : T_xX \to T_{f(x)}Y$ well-def'd linear map.

· Chain rule $dg_{f(x)} \circ df_x = d(g \circ f)_x$ still holds.

For $x \in \partial X$, $T_x(\partial X)$ is codim-1 subsp of T_xX. $d(\partial f)_x = df_x\big|_{T_x(\partial X)}$ $\quad \partial f := f|_{\partial X}$

How to make manifolds with boundary?

e.g. $S^1 \times I$. $\overline{B^n}$. · X mfd w/ bdry ∂X, Y mfd w/o bdry, then $X \times Y$ mfd w/ bdry $\partial X \times Y$. $\dim(X \times Y) = \dim X + \dim Y$

· Y mfd w/o bdry, $\pi : Y \to \mathbb{R}$ has reg value 0, then $\pi^{-1}[0, \infty)$ mfd w/ bdry. (Ex. 11. converse holds.)

· Transversal preimage:

X mfd w/ bdry ∂X, $Z \subseteq Y$ mfds w/o bdry, $f : X \to Y$ s.t. $f \pitchfork Z$, $\partial f \pitchfork Z$ then $f^{-1}(Z)$ mfd w/ bdry $(\partial f)^{-1}(Z)$.

pf) locally $X = \mathbb{H}^n$, $f : \mathbb{R}^n \to Y$. $\pi_n|_{f^{-1}(Z)}$ has reg value 0.

Sard's thm. X mfd w/ bdry ∂X, $f : X \to Y \Rightarrow$ a.e. pt of Y is both reg val of $f, \partial f$.

Ex 7. If X mfd w/ bdry ∂X, then we may define inward tangent space $H_x(X)$ at $x \in \partial X$. Just take param $\phi : \mathbb{H}^k \to X$, $d\phi(\mathbb{H}^k)$.

Ex 8. So for $x \in \partial X$ may define outward unit normal vector $\vec{n}(x)$. This is smooth.

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§2. One-manifolds and some consequences.

<u>Classification of compact one-mfds.</u>

A compact connected 1-mfd w/ bdry is diffeomorphic to S^1 or $[0,1]$.

<u>Cor.</u> A compact 1-mfd w/ bdry has an even number of bdry pts.

<u>No retraction thm.</u> There is no retraction of a compact mfd X onto its bdry ∂X,

$\nexists g: X \to \partial X, \quad g|_{\partial X} = id_{\partial X}$. pf) inv image of reg pt.

<u>Brouwer fixed pt thm.</u> Every $f: \bar{B}^n \to \bar{B}^n$ has a fixed pt. $g(x)$ x $f(x)$

Ex 7. Frobenius thm. Every $A \in [0,\infty)^{n\times n}$ has eigenvector in ~~pos~~ nonneg quadrant.

§3. Transversality

<u>Transversality thm.</u> X mfd w/ bdry ∂X. $S, Z \subseteq Y$ mfd w/o bdry.

$F: X \times S \to Y$ s.t. $F \pitchfork Z$, $\partial F \pitchfork Z$. Then $F_s \pitchfork Z$, $\partial F_s \pitchfork Z$ for a.e. $s \in S$.

pf) $W = F^{-1}(Z)$ mfd w/ bdry $\partial F^{-1}(Z)$. s reg value of $\pi: W \to S$, $\pi: \partial W \to S$ $\to$ $F_s \pitchfork Z$, $\partial F_s \pitchfork Z$.

How do we come up w/ such an F?

<u>Defn</u> $Y \subseteq \mathbb{R}^M$ mfd. <u>Normal space</u> $N_y(Y) = T_y(Y)^\perp$. <u>Normal bundle</u> $N(Y) = \{(y,v) : y \in Y, v \in N_y(Y)\}$.

$N(Y)$ is M-mfd. projection $\sigma: N(Y) \to Y$, $(y,v) \mapsto y$ is submersion. (fact: $A: \mathbb{R}^M \to \mathbb{R}^K$ surj, $\mathbb{R}^K \xrightarrow[\cong]{A^t} \ker(A)^\perp$)

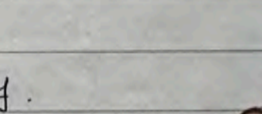
<u>ε-nbhd thm.</u> $Y \subseteq \mathbb{R}^M$ mfd. $\exists\, \varepsilon: Y \to (0,\infty)$ small, $Y^\varepsilon = \bigcup_{y\in Y} B_{\varepsilon(y)}(y)$, $\pi: Y^\varepsilon \to Y$ submersion s.t. $\pi|_Y = id_Y$.

pf) $h: N(Y) \to \mathbb{R}^M$, $(y,v) \mapsto y+v$ is diffeo at Y and identity on Y, so diffeo on nbhd.

<u>Cor</u> $f: X \to Y \subseteq \mathbb{R}^M$ smooth, then $\exists F: X \times B_1^M \to Y$ s.t. $F(\cdot, 0) = f$, $F(x,\cdot)$ is subm.

pf) $F(x,s) = \pi(f(x) + \varepsilon_{f(x)} s)$

<u>Homotopy thm.</u> $f: (X, \partial X) \to Y$, $Z \subseteq Y$ htpc to $g: (X,\partial X) \to Y$ s.t. $g, \partial g \pitchfork Z$.

~~Z clsd needed~~: 

What if we want to presence good parts of f?

<u>Defn</u> $f \pitchfork Z$ on $C \subseteq X$ if $\forall x \in C \cap f^{-1}(Z)$ $df_x(T_xX) + T_{f(x)}Z = T_{f(x)}Y$.

<u>Lem</u> $\underset{clsd}{C} \subseteq \underset{open}{U} \subseteq X$, then $\exists\, \tau: X \to [0,1]$ s.t. $\tau = d\tau = 0$ on nbhd of C, $\tau = 1$ on $X \setminus U$. pf) $C \subseteq int C' \subseteq C' \subseteq U$ P.o.U. wrt $\{U, X \setminus C'\}$ square

<u>Extension thm</u> $f: (X, \partial X) \to Y$, $C \subseteq X$ clsd, $Z \subseteq Y$ <u>clsd</u> submfd, $f \pitchfork Z$ on C, $\partial f \pitchfork Z$ on $C \cap \partial X$.

Then f htpc to $g: (X, \partial X) \to Y$, $g, \partial g \pitchfork Z$, htpy const on C.

pf) extend C to open nbhd. $F: X \times B \to Y$, take $F(x, \tau(x)s)$.

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<u>Cor</u> if $f: X \to Y$, $Z \subseteq Y$ clsd, $\partial f \pitchfork Z$, then may homotope f to $g: X \to Y$ s.t. $\partial f = \partial g$ and $g \pitchfork Z$.

Ex 5. $X, Z \subseteq Y$, X cpt, $\dim X + \dim Z < \dim Y \to X$ can be deformed to not intersect Z.
Ex 7. $X \subseteq \mathbb{R}^N$ submfd, $l \leq N \to$ a.e. v.subsp of dim l of $\mathbb{R}^N$ intersects X transversally.
Ex 12. $Z \subseteq Y \subseteq \mathbb{R}^M$. Normal bundle $N(Z;Y) = \{(z,v) : z \in Z,\ v \in T_z Y \cap N_z Z\}$ is mfd of dim Y.
Ex 16 (Tubular Nbhd Thm) $\exists$ diffeo of open nbhd of Z in $N(Z;Y)$ to open nbhd of Z in Y.
Ex 18. $T(X) \to N(\Delta; X \times X)$, $(x,v) \mapsto ((x,x),(v,-v))$ is diffeo.
So $\exists$ diffeo (nbhd of X in $T(X)$) $\to$ (nbhd of Δ in $X \times X$) ext. $X \to \Delta$.
Ex 19. $N(Z;Y)$ is <u>trivial</u> if $\exists\, N(Z;Y) \xrightarrow{\cong} Z \times \mathbb{R}^k$ fiberwise iso.
$N(Z;Y)$ is <u>locally trivial</u>.
Ex 20. $N(Z;Y)$ trivial $\iff Z$ def'able in Y by indep eqns.

§4. Intersection Theory mod 2

Defn. X and $Z \subseteq Y$ are of <u>complementary dimension</u> if $\dim X = \operatorname{codim}_Y Z$.
S'pse X cpt, $Z \subseteq Y$ closed submfd, compl. dim.
Thm. If $f_0 \overset{\text{htpc}}{,} f_1 : X \to Y$, $f_0, f_1 \pitchfork Z$, then $\# f_0^{-1}(Z) \equiv \# f_1^{-1}(Z) \bmod 2$.
<u>Defn</u>. If $f: X \to Y$, define mod 2 intersection number $I_2(f,Z) = \# g^{-1}(Z) \bmod 2$, where $f \simeq g$ and $g \pitchfork Z$.
· This is a well-defined htpy invariant.
<u>Boundary Thm</u> W cpt w/ bdy X, $f: X \to Y$ extends to $F: W \to Y$, then $I_2(f,Z) = 0$.
Ex 1. If $X \subseteq Y$ then $I_2(X,Z) := I_2(i,Z)$. e.g. $\underset{\text{in } S^1 \times S^1}{I_2}(S^1 \times \{pt\}, \{pt\} \times S^1) = 1$
e.g. $I_2(S^1, S^1) = 1$ in Möb.
Ex 2. If $X \subseteq Y$, $\dim X = \frac{1}{2} \dim Y$, may define self-intersection number $I_2(X,X)$.
Ex 3. If $\dim X = \dim Y$, Y cnnctd, may define mod 2 degree of $f: X \to Y$ as $\deg_2(f) = I_2(f, \{y\})$, $y \in Y$. $\Rightarrow$ htpy invariant, bdy thm.
$\Rightarrow$ roots of fcns: If $W \subseteq \mathbb{C}$ cpt $\overset{2}{\text{mfd}}$ w/ bdy ∂W, if $p: W \to \mathbb{C}$ doesn't vanish on ∂W and $\deg_2(p/|p| : \partial W \to S^1) = 1$, then p has zero in W°.
May prove fund. thm. alg. for odd-deg polyn.

Ex3. $f: X \to Y$, $g: Z \to Y$, $I_2(f,g) := I_2(f \times g, \Delta(Y))$. htpy inv, $I_2(f,g) = I_2(g,f)$, $I_2(f,Z) = I_2(f, i_Z)$, $\overline{I_2(X,Z)} = I_2(Z,X)$

Ex 4. $f: X \to Y$ htpc to const, $\dim X > 0 \to I_2(f, Z) = 0$.

Ex 5. Y contractible, $\dim X > 0 \to I_2(f, Z) = 0$

Ex 6. No cpt mfd besides one-pt sp is contractible.

Ex 7. S^1 not simply connected.

Ex 8. $f: S^1 \to S^1$ lifts to $g: \mathbb{R} \to \mathbb{R}$, $g(2\pi) - g(0) = 2\pi q$. $\deg_2(f) = q \mod 2$.

Ex 14, 15. $X, Z \overset{\text{cpt}}{\subseteq} Y$ cobordant if $W \overset{\text{cpt}}{\subseteq} Y \times I$, $\partial W = X_0 \cup Z_1$.

$C \subseteq Y$ cpt cmpl dim $\to I_2(X, C) = I_2(Z, C)$.

Ex 18, 19. $Z \subseteq Y$ cpt submfd, $\dim Z = \frac{1}{2} \dim Y$. $N(Z;Y)$ trivial $\to I_2(Z,Z) = 0$.

So $N(S^1, \text{Möb})$ not trivial.

§5. Winding Numbers and the Jordan-Brouwer Separation Theorem

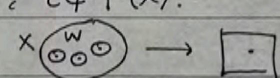
<u>Defn</u> Let X be a compact $(n-1)$-mfd, $f: X \to \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus f(X)$.

Define the mod 2 winding number of f around z

$$W_2(f, z) := \deg_2\left(x \mapsto \frac{f(x) - z}{|f(x) - z|} : X \to S^{n-1} \right).$$

<u>Lemma</u> W cpt w/ bdy X. $F: W \to \mathbb{R}^n$, z reg val of F, $z \notin F(X)$.

Then $W_2(\partial F, z) \equiv \# F^{-1}(z) \mod 2$.



<u>Jordan-Brouwer Separation Thm.</u>

Let X be a compact connected hypersurface of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus X$ has two components,

inside $D_1 = \{z : W_2(X, z) = 1\}$, outside $D_0 = \{z : W_2(X, z) = 0\}$.

D_1 is cpt mfd w/ bdy X.

Given $z \in \mathbb{R}^n \setminus X$, take ray $r = \{z + t\vec{v} : t \geq 0\}$ transversal to X. Then $z \in D_{\# r^{-1}(X) \bmod 2}$.

pf) Step 1. Every $z \in \mathbb{R}^n \setminus X$ is path-con in $\mathbb{R}^n \setminus X$ to a nbhd of $x \in X$ in $\mathbb{R}^n \setminus X$. So ≤ 2 comps.

Step 2. D_0, D_1 open partition of $\mathbb{R}^n \setminus X$.

Step 2. ray r transversal to X iff $\vec{v}$ reg val of $\widehat{f - z}$. So holds, ≥ 2 comps.

Winding # 0 if z far away

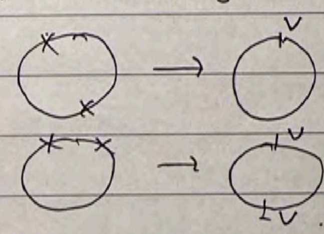
§6. The Borsuk-Ulam Theorem

Thm. A map $f: S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ that is symmetric $(f(-x) = -f(x))$ has $W_2(f, 0) = 1$.

$\Leftrightarrow$ A map $f: S^k \to S^k$ that is symmetric has $\deg_2(f) = 1$.

pf) $\underbrace{S^{k-1}}_{\text{equator}} \subseteq \underbrace{S^k_+}_{\text{upper hemi}} \subseteq S^k$, $f' \subseteq f_+ \subseteq f$. Take $v \in S^k$ reg val of f not in image(f').

$$\deg_2(f) = \# f^{-1}(v) = \frac{1}{2} \# f^{-1}(\pm v) = \# f_+^{-1}(\pm v)$$



Take projection $\pi_v: S^k \to \mathbb{R}^k$.

$$\# (\pi_v \circ f_+)^{-1}(\underbrace{0}_{\text{reg val}}) = W_2(\pi_v \circ f', 0) = 1$$

by induction.

Cor. Any symmetric map $f: S^k \to \mathbb{R}^{k+1}$ intersects any ray thru the origin.

Any symmetric map $f: S^k \to \mathbb{R}^k$ has 0 in its image.

Any map $g: S^k \to \mathbb{R}^k$ has antipodal $p, -p$ w/ same value.

cf) Lovasz's proof of the Kneser conjecture.

"Using the Borsuk-Ulam thm"

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Kneser's Conjecture.

Recall the Borsuk-Ulam thm:

If $f: S^k \to \mathbb{R}^k$ smooth, $\exists p \in S^k$ $f(p) = f(-p)$.

We have a conti version:

Conti Borsuk-Ulam thm.

If $f: S^k \to \mathbb{R}^k$ conti, $\exists p \in S^k$ $f(p) = f(-p)$.

pf) S'pse not, then

$$\inf_{p \in S^k} |f(p) - f(-p)| =: \varepsilon > 0 \quad \text{by cptness. \& continuity.}$$

By Stone-Weierstrass, $\exists$ polyn fcn $g: S^k \to \mathbb{R}^k$ s.t.

$$\sup_{p \in S^k} |f(p) - g(p)| < \frac{\varepsilon}{2}.$$

But then $\forall p \in S^k$

$$|g(p) - g(-p)| \geq |f(p) - f(-p)| - |f(p) - g(p)| - |f(-p) - g(-p)| > 0,$$

contradicting smooth Borsuk-Ulam. □

Borsuk-Ulam thm (covering ver.)

If $U_1, \cdots, U_{k+1} \subseteq S^k$ open s.t. $U_1 \cup \cdots \cup U_{k+1} = S^k$,

then $\exists i \in \{1, \cdots, k+1\}$ $\exists p \in S^k$ $p \in U_i$ and $-p \in U_i$.

pf) The function $S^k \to \mathbb{R}$, $x \mapsto d(x, S^k \setminus U_i)$ is continuous with

$$d(x, S^k \setminus U_i) \neq 0 \iff x \in U_i.$$

Consider the function $f: S^k \to \mathbb{R}^k$,

$$f(x) = (d(x, S^k \setminus U_1), \cdots, d(x, S^k \setminus U_k)), \qquad x \in S^k.$$

Since f is conti, by conti Borsuk-Ulam, $\exists p \in S^k$ $f(p) = f(-p)$.

If $f(p) \neq 0$, $\exists i_{\in\{1,\dots,k\}}$ $d(p, S^k \setminus U_i) = d(-p, S^k \setminus U_i) \neq 0 \Rightarrow p, -p \in U_i$.

If $f(p) = f(-p) = 0$, then $\forall i \in \{1, \cdots, k\}$ $d(p, S^k \setminus U_i) = d(-p, S^k \setminus U_i) = 0 \Rightarrow p, -p \notin U_i$.

Since $U_1, \cdots, U_{k+1}$ cover S^k, $p, -p \in U_{k+1}$.

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Borsuk-Ulam thm (covering ver, closed sets allowed.)

If $U_1, \cdots, U_k \subseteq S^k$ open, $F \subseteq S^k$ closed s.t. $U_1 \cup \cdots \cup U_k \cup F = S^k$,

then $\exists p \in S^k$ s.t. either $\exists i \in \{1, \cdots, k\}$ $p, -p \in U_i$ or $p, -p \in F$.

pf) S'pse F contains no antipodal pairs of pts, i.e. $\forall p \in F$ $-p \notin F$.

Then

$$\inf_{q_1, q_2 \in F} |q_1 + q_2| = \varepsilon > 0, \text{ by cptness of } F \text{ and continuity.}$$

Define

$$U_{k+1} := \text{Nbhd}_{\varepsilon/2}(F) = \{p \in S^k \mid d(p, F) < \tfrac{\varepsilon}{2}\}$$

$$= \bigcup_{q \in F} B_{\varepsilon/2}(q).$$

Then $F \subset U_{k+1}$, and U_{k+1} also contains no antipodal pairs of pts :

$$p_1, p_2 \in U_{k+1} \rightarrow \exists q_1, q_2 \in F \quad |p_1 - q_1| < \varepsilon/2, \; |p_2 - q_2| < \varepsilon/2$$

$$\rightarrow |p_1 + p_2| \geq |q_1 + q_2| - |p_1 - q_1| - |p_2 - q_2| > 0$$

$$\rightarrow p_1 \neq -p_2.$$

Now $U_1, \cdots, U_k$ and U_{k+1} form an open cover of S^k, so by the previous thm, one of them must contain an antipodal pair of pts, but since this is not the case for U_{k+1}, it must be the case for one of $U_1, \cdots, U_k$.

Rmk) One can use the above trick to replace any number of open sets in the covering into closed sets, to the effect of proving that, any covering of S^k by $(k+1)$ sets, where each set is either closed or open, contains an elt containing an antipodal pair of pts.

morning glory

Defn. A finite graph is a pair $G=(V,E)$, where V is a finite set and E is a collection of unordered pairs of distinct elts in V.

For $k\in\mathbb{Z}_{>0}$, a (vertex) k-coloring of G is a function $c: V\to\{1,\cdots,k\}$ s.t. whenever $\{u,v\}\in E$, $c(u)\neq c(v)$.

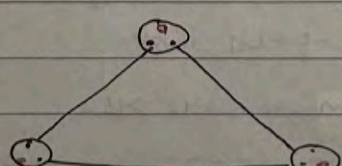
The chromatic number $\chi(G)$ of G is the smallest $k\in\mathbb{Z}_{>0}$ s.t. G admits a k-coloring.

Given a graph G, it is an interesting and often hard question to compute $\chi(G)$. For example, the four color thm asserts that any planar graph (i.e., $V\subseteq\mathbb{R}^2$, E is s.t. when each edge $\{u,v\}$ is realized as a straight line segment from u to v, no two edges intersect except possibly at their endpoints) has chromatic number ≤ 4.

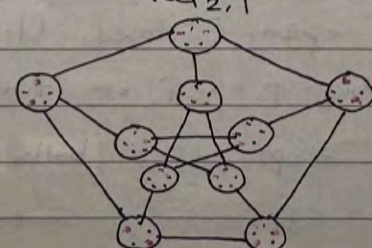
Defn. For $n,k\in\mathbb{Z}_{>0}$, the Kneser graph $KG_{n,k}$ is the graph

$$V_{n,k}=\{\text{subsets of } \{1,\cdots,2n+k\} \text{ of size } n\}$$
$$E_{n,k}=\{\{u,v\}: u,v\in V,\ u\cap v=\phi\}.$$

$KG_{1,1}$



$KG_{2,1}$



a.k.a. Petersen graph.

Conjecture (Kneser, 1956) $\chi(KG_{n,k})=k+2$.

Rmk) $\chi(KG_{n,k})\leq k+2$ is easy: denoting $K_i=\{u\in V_{n,k}: \min u=i\}$,

$$K_1,\ K_2,\ \cdots,\ K_{k+1},\ K_{k+2}\cup\cdots\cup K_{k+n+1}$$

is a $(k+2)$-coloring, i.e., any two u,v in the same color class intersect.

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So, the hard part is showing the lower bound, i.e., a $(k+1)$-coloring is impossible.

Thm (Lovász 1978). Kneser's conjecture is true.

Lovász's proof uses algebraic topology, especially the Borsuk-Ulam thm. This has been shortened by Bárány (1978) and Greene (2002), whose presentation we follow.

For $a \in S^{k+1}$, denote

$H(a) = \{x \in S^{k+1} : a \cdot x > 0\}$, open hemisphere centered at a,

$S(a) = \{x \in S^{k+1} : a \cdot x = 0\}$, great k-sphere perp. to a.

Place $2n+k$ points on S^{k+1} in general position, i.e., no $k+2$ points lie on a great k-sphere. Partition the n-pt subsets into $A_1, \dots, A_{k+1}$. Denote

$U_i = \{a \in S^{k+1} : \exists u \in A_i \ \ u \subseteq H(a)\}$, $\quad i = 1, \dots, k+1$,

$F = S^{k+1} \setminus (U_1 \cup \dots \cup U_{k+1})$.

$\boxed{\text{Need to show: } \exists i \in \{1, \dots, k+1\} \ \exists u, v \in A_i \ \ u \cap v = \phi.}$

Then $U_1, \dots, U_{k+1}$ open, F closed, $U_1 \cup \dots \cup U_{k+1} \cup F = S^{k+1}$, so by Borsuk-Ulam, $\exists p \in S^{k+1}$ s.t. either $p, -p \in F$ or $\exists i \in \{1, \dots, k+1\}$ $p, -p \in U_i$. But

$p, -p \in F \Rightarrow H(p), H(-p)$ each contains at most $(n-1)$ pts

$\Rightarrow S(p)$ contains at least $2n+k-2(n-1) = k+2$ pts

$\Rightarrow \cancel{\ast}$

so $\exists i \in \{1, \dots, k+1\}$ $\exists u, v \in A_i$ s.t. $u \subseteq H(p)$, $v \subseteq H(-p)$ so that $u \cap v = \phi$.

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Chp 3. Oriented Intersection Theory.

§3.1 Motivation.

May enhance intersection numbers by considering orientations.

§3.2 Orientation.

<u>Defn</u> Two ordered bases β and β' of a f.d.v.s. V are <u>equivalently oriented</u> if the isomorphism $A: V \to V$ s.t. $\beta' = A\beta$ (i.e. $\beta = (v_1, \cdots, v_k)$, $\beta' = (v_1', \cdots, v_k')$, $v_i' = Av_i$) has $\det A > 0$.

(Recall: $\det A$ is def'd indep. of choice of basis.)

This is an equivalence relation, and if $\dim V \geq 1$, there are exactly two equivalence classes.

If $\dim V = 0$, there is only one basis ϕ, and the zero map $0: V \to V$ is def'd to have det 1.

An <u>orientation</u> of V is a choice to assign $+1$ to one equiv class and -1 to the other (if it exists). So we have

$$\mathrm{sgn}_V : (\text{bases of } V) \to \{\pm 1\}$$

s.t.

$$\mathrm{sgn}_V(A\beta) = \mathrm{sgn} \det A \cdot \mathrm{sgn}_V \beta.$$

We give $\mathbb{R}^k$ the standard ori: $(e_1, \cdots, e_k)$ is pos, or $\mathrm{sgn}_{\mathbb{R}^k}(v_1, \cdots, v_k) = \mathrm{sgn} \det(v_1, \cdots, v_k)$.

The terms "positively oriented basis" and "negatively oriented basis" are defined in the obvious way.

V is <u>oriented</u> if it is given an orientation.

Defn. An isom $A: V \to W$ between oriented v.sps is orientation preserving (reversing) if given a positively oriented basis β of V, $A\beta$ is positively (negatively) oriented in W.

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Well-defdness:

$$\begin{tikzcd} V \arrow[r, "A", "\approx"'] \arrow[d, "B"', "\wr\wr"] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, "ABA^{-1}", "\wr\wr"'] \\ V \arrow[r, "\approx", "A"'] & W \end{tikzcd} \qquad \begin{tikzcd} \beta \arrow[r, mapsto] \arrow[d, mapsto] & A\beta \arrow[d, mapsto] \\ B\beta = \beta' \arrow[r, mapsto] & A\beta' = (ABA^{-1})\beta \end{tikzcd}$$

$\det(ABA^{-1}) = \det B$

Alternatively, $A: V \to W$ preserves/reverses ori. if the const fcn

$$\operatorname{sgn}_W(A\,\cdot) \big/ \operatorname{sgn}_V(\cdot) = 1 \text{ or } -1.$$

Defn: An orientation of a mfd (w/ or w/o bdry) X is a choice of orientation for every tangent space T_xX, $x \in X$, that is continuous:

$\forall x \in X$ $\exists$ param $h: U(\subseteq \mathbb{H}^k) \to V(\underset{\ni x}{\subseteq X})$ s.t. $\forall u \in U$ $dh_u: \underset{\substack{\uparrow \\ \text{w/ std ori}}}{\mathbb{R}^k} \to T_{h(u)}X$ is orientation preserving.

- If X is a 0-mfd, an ori. of X is equivalent to a map $X \to \{1, -1\}$.

Defn. A mfd X is orientable if it can be given an orientation.

cf) Some mfds are nonorientable, e.g. Möbius strip.

$\boxed{\text{Def. If } X \subseteq Y,\ \dim X = \dim Y,\ Y \text{ oriented, } X \text{ inherits an ori from } Y.}$

A mfd X (w/ bdry) is oriented if it is given an orientation.

If X is oriented, denote by $-X$ the mfd X where each T_xX is given the opposite orientation than that of X.

Prop. If X is a connected oriented mfd, then X and $(-X)$ are the only two possible orientations of X.

pf) If $x \in X$ and $h: U(\subseteq \mathbb{H}^k) \to V(\underset{\ni x}{\subseteq X})$ and $h': U'(\subseteq \mathbb{H}^k) \to V'(\underset{\ni x}{\subseteq X})$ are two params around x, then $\operatorname{sgn} \det d(h' \circ h^{-1}): h'^{-1}(V' \cap V) \to \{\pm 1\}$ is conti hence const in a nbhd of $h'^{-1}(x)$.

So if h pres. ori. for 1st ori, h' pres. ori for 2nd ori, either the two ori-s agree or disagree in a nbhd of x.

$W = \{x \in X \mid \text{two oris agree at } x\}$ open

$X \setminus W = \{x \in X \mid \text{two oris disagree at } x\}$ open

$\Rightarrow$ either $W = X$ or $W = \phi$.

w/ pos. ori. bases α and β,
$\alpha = (v_1, \dots, v_k)$, $\beta = (w_1, \dots, w_l)$

Defn. Given oriented f.d. v.s. V, W, the <u>product orientation</u> on $V \times W$ is given by setting $(\alpha \times 0, 0 \times \beta) = (v_1 \times 0, \dots, v_k \times 0, 0 \times w_1, \dots, 0 \times w_l)$ a pos. basis.

well-def'dness: if $A: V \to V$, $B: W \to W$ iso, then $A \times B : V \times W \to V \times W$ def'd by
$(A \times B)(v, w) = (Av, Bw)$
has $\det(A \times B) = \det A \cdot \det B$. So if V has pos. ori bases α and α', W has pos ori bases β and β', $A\alpha = \alpha'$, $B\beta = \beta'$, then $\operatorname{sgn} \det A > 0$, $\operatorname{sgn} \det B > 0$,

$$(A \times B)(\alpha \times 0, 0 \times \beta) = (\alpha' \times 0, 0 \times \beta'),$$
$$\operatorname{sgn} \det (A \times B) = \operatorname{sgn} \det A \operatorname{sgn} \det B > 0.$$

Defn. If X, Y oriented mfds, at least one of which is bdyless, the <u>product orientation</u> of $X \times Y$ gives $T_{(x,y)}(X \times Y) = T_x X \times T_y Y$, $(x,y) \in X \times Y$, the product orientation from $T_x X$ and $T_y Y$.

<u>check</u> this is a conti. ori.

Defn. Let X be an oriented mfd w/ bdry ∂X.
~~exercise 2.1.8~~
(For $x \in \partial X$ denote by $\vec{n}_x \in T_x X \cap (T_x(\partial X))^\perp$ the outward unit normal vector.)
The <u>boundary orientation</u> on ∂X is def'd as

$$\operatorname{sgn}_{T_x(\partial X)}(v_1, \dots, v_{k-1}) = \operatorname{sgn}_{T_x X}(n_x, v_1, \dots, v_{k-1}), \quad x \in X$$

whenever $v_1, \dots, v_{k-1}$ is a basis of $T_x(\partial X)$

<u>Check</u> well-def'dness, continuity.

ex) $\overline{\mathbb{D}_1} \subseteq \mathbb{R}^2$ inherits std ori, $\partial \overline{\mathbb{D}_1} = S^1$ bdy ori is counterclockwise ori.

ex) $[0,1] \subseteq \mathbb{R}$ inherits std ori, $\partial [0,1] = \{1\} - \{0\}$ bdy ori.

ex) $(X, \partial X)$, (Y, ϕ) oriented $\Rightarrow$ $\partial(X \times Y) = \partial X \times Y$ have same ori.

ex) $\partial([0,1] \times X) = \{1\} \times X - \{0\} \times X$.

<u>Obs</u> The sum of the ori. numbers at the bdy pts of a compact oriented 1-dim mfd w/ bdy is 0.

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Defn. If $V = U \oplus W$, all f.d.v.s., α basis of U, β basis of W, two of U, V, W oriented. Then the relation

$$\mathrm{sgn}_V(\alpha, \beta) = \mathrm{sgn}_U(\alpha)\, \mathrm{sgn}_W(\beta)$$

allows us to define a <u>direct sum orientation</u> on the other.

<u>check</u> well-def'd.

also, $\mathrm{sgn}_{U \oplus W} = (-1)^{\dim U \dim W} \mathrm{sgn}_{W \oplus U}$.

Defn. Let U, V, W be f.d.v.s. and let

(exercise 3.2.3)

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "A"] & V \arrow[r, "B"] & W \arrow[r] \arrow[l, dashed, bend left, "h"] & 0 \end{tikzcd}$$

be a short exact sequence. ($V_1 \xrightarrow{A_1} V_2 \cdots \to V_n \xrightarrow{A_n} V_{n+1}$ is <u>exact</u> if $\mathrm{image}(A_i) = \ker(A_{i+1})\ \forall i$. So this means $\mathrm{image}(A) = \ker(B)$, A inj, B surj. So $W \simeq V/A(U)$). This SES is <u>split</u> : $\exists\, h: W \to V$ s.t. $B \circ h = \mathrm{id}_W$, and hence $V = h(W) \oplus A(U)$

If α basis of U, β basis of W, the relation

$$\mathrm{sgn}_V(h\beta ; A\alpha) = \mathrm{sgn}_U(\alpha)\, \mathrm{sgn}_W(\beta)$$

allows us to define a <u>short exact sequence orientation</u> from two among U, V, W on the third.

<u>Check</u> well-def'd: indep of choice of α, β, h.

ex) If $U \leq V$, we have SES

$$0 \to U \xrightarrow{i} V \xrightarrow{\pi} V/U \to 0.$$

If $\pi: V \to W$ surj, we have SES

$$0 \to \ker \pi \xrightarrow{i} V \xrightarrow{\pi} W \to 0.$$

Defn. If $f: X \to Y$ has reg val $y \in Y$, X, Y oriented,
then give $f^{-1}(y)$ the <u>preimage orientation</u> via the
short exact sequence orientation:

$$0 \to T_x f^{-1}(x) = \ker(df_x) \xrightarrow{i} T_x X \xrightarrow{df_x} T_y Y \to 0.$$

(In case $\{y\}$ also oriented, multiply above ori by $\operatorname{sgn}(y)$.)

ex) bdry ori on ∂X is given by preim. ori from $\pi: X \to \mathbb{R}$.

Def/Prop. Let U, V, V' be oriented f.d.v.s., $U \leq V$ and $A: V' \to V$ s.t.
$\operatorname{image}(A) + U = V$.
Then there is an iso $\bar{A}: V'/A^{-1}(U) \to V/U$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & A^{-1}(U) \arrow[r, "i'"] \arrow[d, "A|_{A^{-1}(U)}"'] \arrow[dr, phantom, "\circlearrowright"] & V' \arrow[r, "\pi'"] \arrow[d, "A"] \arrow[dr, phantom, "\circlearrowright"] & V'/A^{-1}(U) \arrow[r] \arrow[d, "\bar{A}", "\wr"'] & 0 \\
0 \arrow[r] & U \arrow[r, "i"'] & V \arrow[r, "\pi"'] & V/U \arrow[r] & 0
\end{tikzcd}$$

(A induces a chain map)

Give V/U the SES ori from U and V. Give $V'/A^{-1}(U)$ the ori
induced by $\bar{A}$ from V/U. Give $A^{-1}(U)$ the SES ori from V' and $V'/A^{-1}(U)$.
This is called the <u>preimage orientation</u>.

Def. Let $f: (X, \partial X) \to Y \supseteq Z$, X, Y, Z oriented, $f \pitchfork Z$, $\partial f \pitchfork Z$.
Give $f^{-1}(Z)$ the <u>preimage ori</u> from $U = T_{f(x)} Z \leq V = T_{f(x)} Y$,
$A = df_x : T_x X \to T_{f(x)} Y$:

$$\begin{tikzcd}
0 \arrow[r] & T_x f^{-1}(Z) \arrow[r, "i"] \arrow[d, "df_x|"'] \arrow[dr, phantom, "\circlearrowright"] & T_x X \arrow[r, "\pi"] \arrow[d, "df_x"] \arrow[dr, phantom, "\circlearrowright"] & T_x X / T_x f^{-1}(Z) \arrow[r] \arrow[d, "\overline{df_x}"] & 0 \\
0 \arrow[r] & T_{f(x)} Z \arrow[r, "i"'] & T_{f(x)} Y \arrow[r, "\pi"'] & T_{f(x)} Y / T_{f(x)} Z \arrow[r] & 0
\end{tikzcd}$$

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Prop. $\partial[f^{-1}(Z)] = (-1)^{\operatorname{codim} Z}\,(\partial f)^{-1}(Z)$.

pf)

$$\begin{tikzcd}
0 \arrow[r] & T_x(\partial f)^{-1}(Z) \arrow[r] \arrow[d, "i"] & T_x(\partial X) \arrow[r] \arrow[d, "i"] & H_1 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & T_x f^{-1}(Z) \arrow[r] \arrow[d, "df_x"] & T_x X \arrow[r] \arrow[d, "df_x"] & H_2 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & T_{f(x)} Z \arrow[r] & T_{f(x)} Y \arrow[r] & H_3 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
T_x \partial(f^{-1}(Z)) \arrow[r, "i"] \arrow[d] & T_x(\partial X) \arrow[d] \\
T_x f^{-1}(Z) \arrow[r] \arrow[d, "p"] & T_x X \arrow[d, "p"] \\
\mathbb{R} \arrow[r, "\approx"] \arrow[d] & \mathbb{R} \arrow[d] \\
0 & 0
\end{tikzcd}$$

$T_x X = \mathbb{R} \oplus T_x(\partial X)$

$T_x(\partial X) = H_1 \oplus T_x(\partial f)^{-1}(Z)$

$\Rightarrow T_x X = \mathbb{R} \oplus H \oplus T_x(\partial f)^{-1}(Z)$

$T_x X = H_2 \oplus T_x\, f^{-1}(Z)$

$T_x f^{-1}(Z) = \mathbb{R} \oplus T_x\, \partial(f^{-1}(Z))$

$\Rightarrow T_x X = H \oplus \mathbb{R} \oplus T_x\, \partial(f^{-1}(Z))$.

§3.3 Oriented Intersection Number.

1. Basic defn

<u>Defn</u>. $f: X \to Y$ and Z are <u>appropriate for intersection theory</u> if X, Y, Z bdyless, X cpt, Z clsd submfd of Y, $\dim X + \dim Z = \dim Y$, X, Y, Z oriented.

<u>Defn</u> If $f: X \to Y$ and Z app. for int. theory and $f \pitchfork Z$, then the <u>intersection number</u> $I(f, Z)$ is defined as

$$I(f,Z) = \sum_{x \in f^{-1}(Z)} (\text{ori. of } f^{-1}(Z), \text{ w/ preim. ori, at } x).$$

Rmk. · $f^{-1}(Z)$, being a cpt mfd, consists of finitely many pts. · $I(f,Z) = I_2(f,Z) \bmod 2$

· (ori. of $f^{-1}(Z)$ at x) $= \operatorname{sgn}_{T_{f(x)}Y}\left(df_x(T_x X) \oplus T_{f(x)} Z \right)$

Prop. $X = \partial W$, W cpt ori, $f: X \to Y$ ($f \pitchfork Z$) ext to $F: W \to Y$, then $I(f, Z) = 0$.

pf) By ext thm, may assume $F \pitchfork Z$. Then $f^{-1}(Z) = (-1)^{\operatorname{codim} Z}\, \partial(F^{-1}(Z))$.

morning glory

Prop. $f, g : X \to Y$, $f, g \pitchfork Z$ htpc $\Rightarrow I(f,Z) = I(g,Z)$.

pf) $W = I \times X$.

Defn. $f : X \to Y$, Z app for int. theory $\to I(f,Z) := I(g,Z)$,
where $g : X \to Y$, $f \sim g$, $g \pitchfork Z$.

2. Degree

Def. X, Y oriented bdyless, X cpt, Y cnnctd, $\dim X = \dim Y$. ~~Then~~ $f : X \to Y$.

$\deg(f) := I(f, \{\underset{}{pt}\overset{\in Y}{}\})$.

Rmk. - Well-defined htpy invariant.

· Let $y \in Y$ be a regular value of f. Then $f^{-1}(y)$ finite,

$$\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sgn}_{T_yY}(df_x(T_xX)).$$

ex. $S^1 \to S^1$, $z \mapsto z^m$ has deg m, $m \in \mathbb{Z}$. So the htpy classes of C^∞-maps $S^1 \to S^1$ are infinite (actually "$\mathbb{Z} \leq \pi_1(S^1)$").

Prop. $\deg(X \xrightarrow{f} Y \xrightarrow{g} Z) = \deg f \cdot \deg g$.

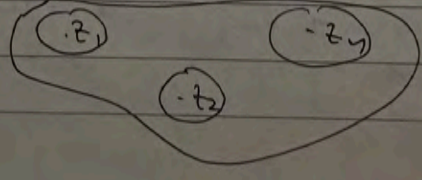
Prop. If $X = \partial W$, W cpt oriented, $f : X \to Y$ ext to $F : W \to Y$, then $\deg(f) = 0$.

FTA. Every nonconst $\mathbb{C}$-polyn p has a zero in $\mathbb{C}$.

pf) If not, then for large r, $p/|p| : \partial B_R \to \partial D$ is htpc to $z \mapsto (z/R)^m$, $m =$ polyn deg of p, which has deg m.

Prop. Let $p(z) \in \mathbb{C}[z]$, W cpt 2-submfd of $\mathbb{C}$ w/ bdy ∂W not containing zeroes of p. Then

$$\deg\left(p/|p| : \partial W \to S^1\right) = \sum_{\substack{z_0 \text{ zero of } p \\ \text{in } W}} (\text{multiplicity of } z_0 \text{ w.r.t. } p)$$

pf) 

$$\deg(p/|p| : \partial W \to S^1) = \sum_{i=1}^{n} \deg(p/|p| : \partial D_{\varepsilon_i}(z_i) \to S^1)$$
$$= \sum_{i=1}^{n} \deg\left(\left(\frac{z - z_i}{|z - z_i|}\right)^{l_i} : \partial D_{\varepsilon_i}(z_i) \to S^1\right)$$

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3. X submfd of Y?
May define $I(X,Z) = I(i_X : X \hookrightarrow Y, Z)$. If $X \pitchfork Z$ in Y then

$$I(X,Z) = \sum_{p \in X \cap Z} \operatorname{sgn}_{T_pY}(T_pX \oplus T_pZ),$$

and

$$I(Z,X) = (-1)^{\dim X \dim Z} I(X,Z). \qquad (\text{assuming } Z \text{ cpt})$$

But what if $X \not\pitchfork Z$? Above requires deformation in Z as well.

Let $X, Y, \cancel{\text{cpt}}\ Z$ ori, X, Z cpt, $\dim X + \dim Z = \dim Y$.

~~Prop~~ **Def.** Let $f: X \to Y$, $g: Z \to Y$. Then

$$f \pitchfork g :\Leftrightarrow \forall x \in X\ z \in Z \text{ s.t. } f(x) = g(z) =: y,$$
$$df_x(T_xX) + dg_z(T_zZ) = T_yY.$$

$\cancel{\Rightarrow I(f,g) = \sum_{f(x)=g(z)}}$

~~Prop~~ **Lem.** If $U, W \leq V$, then $U \oplus W = V$ iff $(U \times W) \oplus \Delta(V) = V \times V$.
If U, W ori, $V = U \oplus W$ dir. sum. ori, then

$$V \times V = (-1)^{\dim W} (U \times W) \oplus \Delta(V).$$

Prop. $f \pitchfork g \Leftrightarrow (f \times g) \pitchfork \Delta(Y)$ in $Y \times Y$

$$I(f,g) := \sum_{f(x)=g(z)} \operatorname{sgn}_{T_yY}\left(df_x(T_xX) \oplus dg_z(T_zZ)\right)$$
$$= (-1)^{\dim Z} I(f \times g, \Delta(Y)).$$

Hence

<u>Defn</u> $X \xrightarrow{f} Y$, $Z \xrightarrow{g} Y$,

$$I(f,g) := (-1)^{\dim Z} I(f \times g, \Delta).$$

Prop. $f_0 \sim f_1$, $g_0 \sim g_1 \Rightarrow I(f_0, g_0) = I(f_1, g_1)$ pf) $f_0 \times g_0 \sim f_1 \times g_1$

Prop. $I(f,g) = (-1)^{\dim X \dim Z} I(g,f)$.

pf)
$$\begin{tikzcd}
X \times Z \arrow[r, "f \times g"] \arrow[d, "\text{flip}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \times Y \arrow[d, "\text{flip}"] \arrow[dr, phantom, "\circlearrowleft"] & \Delta(Y) \arrow[l, hook'] \arrow[d, "\text{id}"] \\
Z \times X \arrow[r, "g \times f"'] & Y \times Y & \Delta(Y) \arrow[l, hook']
\end{tikzcd}$$

$I(f \times g, \Delta) = (-1)^{\dim X \dim Z} (-1)^{(\dim Y)^2} I(g \times f, \Delta)$

cf) book proof assumes $f \pitchfork g$.

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Obs $i: Z \hookrightarrow Y$ submfd $\Rightarrow I(f, i) = I(f, Z)$.

Cor $X, Z \subseteq Y$ $\Rightarrow I(X, Z) = (-1)^{\dim X \dim Z} I(Z, X)$.

Lem $X \subseteq Y$, X cpt ori'able, $\dim X = \frac{1}{2}\dim Y$ odd, $I_2(X, X) = 1$
$\Rightarrow$ Y not ori'able.

Cor Möb not ori'able.

Defn Euler characteristic of cpt ori mfd Y is
$$\chi(Y) := I(\Delta(Y), \Delta(Y)) \text{ in } Y \times Y.$$

Prop Y odd-dim $\Rightarrow \chi(Y) = 0$.

§3.4 Lefschetz Fixed Point Theory.

Recall: X cpt mfd, $f: X \to X$, $F: X \to X \times X$, $x \mapsto (x, f(x))$.

- x fixed pt of f $:\Leftrightarrow f(x) = x \Leftrightarrow F(x) \in \Delta(X)$
- x Lefschetz fxd pt of f $:\Leftrightarrow df_x$ doesn't have e-val 1 $\Leftrightarrow F \pitchfork \Delta(X)$ at x.
- f Lefschetz $:\Leftrightarrow \forall$ fxd pt Lef. $\Leftrightarrow F \pitchfork \Delta(X)$.
  $\Rightarrow$ fxd pts are finite.

Defn If X cpt ori, $f: X \to X$, the global Lefschetz number of f
$$L(f) := I(\Delta(X), \mathrm{graph}(f)) \quad \left(= (-1)^{\dim X} I(F, \Delta(X))\right).$$

Defn/obs If x Lef fxd pt of f,
$$L_x(f) := (\text{intersection number of } \Delta(X) \text{ and graph}(f))$$
$$= \mathrm{sgn} \det (df_x - I_{T_xX}).$$

pf) $\mathrm{sgn}_{T_xX \times T_xX}(\Delta(T_xX) \oplus \mathrm{graph}(df_x))$
$= \mathrm{sgn}_{T_xX \times T_xX}((v_1, v_1), \dots, (v_k, v_k), (v_1, df_x(v_1)), \dots, (v_k, df_x(v_k)))$
$= \mathrm{sgn} \det \begin{pmatrix} I_k & I_k \\ I_k & [df_x] \end{pmatrix} = \mathrm{sgn} \det \begin{pmatrix} I_k & 0 \\ I_k & df_x - I_k \end{pmatrix} = \mathrm{sgn} \det (df_x - I_k)$

Cor If f Lefschetz, $L(f) = \sum_{f(x)=x} L_x(f) = \sum_{f(x)=x} \mathrm{sgn} \det (df_x - I_k)$.

Prop. · $L(f)$ is a htpy invariant.

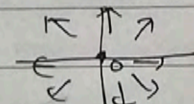
· $L(\mathrm{id}_X) = \chi(X)$.

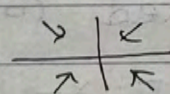
· (Smooth Lefschetz Fixed-pt thm).
If f has no fixed pt, $L(f) = 0$. If $L(f) \neq 0$, f has a fixed pt.

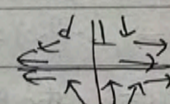
· If $f \sim \mathrm{id}_X$, then $L(f) = \chi(X)$. So if $f \sim \mathrm{id}_X$ and f has no fixed pts, then $\chi(X) = 0$.

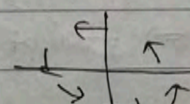
Ex. Let x be a fixed pt of f, and choose a param. ~~near~~ $\psi: U \to X$, $0 \mapsto x$, under which $(\psi^{-1} \circ f \circ \psi)(y) = Ay + o(|y|)$. If A doesn't have e-val 1, then x is Lefschetz, and $L_x(f) = \operatorname{sgn} \det(A - I)$.

e.g. $k=2$,

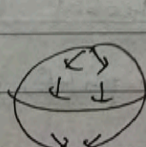
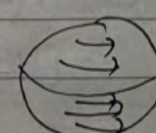
$A = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}$ $\alpha_1, \alpha_2 > 1$.  $L_x(f) = 1$

$A = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}$ $\alpha_1, \alpha_2 < 1$  $L_x(f) = 1$

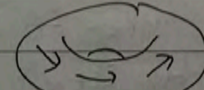
$A = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}$ $\alpha_1 > 1 > \alpha_2$  $L_x(f) = -1$.

~~ex. $\chi(S^2)$~~ $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$  $L_x(f) = 1$.

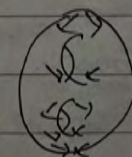
ex. $\chi(S^2) = 2$.

pf) $x \mapsto \dfrac{x - te_3}{|x - te_3|}$  or $x \mapsto Ax$, A rot. 

ex) $\chi(T^2) = 0$

pf) $(x, y) \mapsto (e^{i\theta}x, y)$ 

ex) $\chi(S_g) = 2 - 2g$.

pf) $\pi(x - te_3)$. 

Classification of 2-mfds: cpt ori bdyless is S_g.

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Prop. Every $f: X \to X$ htpc to a Lefschetz map.

pf) May find $F: X \times B \to X$ s.t. $F(x,0) = f(x)$, and $F|_{\{x\} \times B}$ is subm.

Define $G: X \times B \to X \times X$, $G(x,s) = (x, F(x,s))$ is a subm,

so $\exists s$ $G|_{X \times \{s\}} \pitchfork \Delta(X)$. Then ~~$F_s \pitchfork \Delta(X)$~~, F_s Lefschetz.

This is not constructive, i.e., we don't know much about the Lefschetz map htpc to f, especially its fixed pts. E.g., if your map locally looks like $z \mapsto z + z^m$, $z \in \mathbb{C}$, ~~what~~ how does a ~~locally~~ htpc Lefschetz map look like?

Splitting Proposition. Let $x \in X$ be an isolated fixed pt of $f: X \to X$, say $x \in U$ w/ no other fxd pt of f in U. There is a htpy f_t cptly spt'd in U s.t. f_1 only has Lefschetz fixed pts in U.

pf) By parametrizing, may assume $x = 0 \in U \subseteq \mathbb{R}^k$ and $f: U \to \mathbb{R}^k$, only fxd pt 0.

If $-a \in \mathbb{R}^k$ reg pt of $x \mapsto f(x) - x$, then

$$f(x) - x = -a \ (\Leftrightarrow f(x) + a = x) \Rightarrow df_x - I_k \text{ nonsingular.}$$

Take $\rho: \mathbb{R}^k \to [0,1]$ smooth, $=1$ ~~near~~ on nbhd V of 0 and $\operatorname{spt} \rho \subseteq U$.

Let $\varepsilon = \min_{x \in \operatorname{spt}\rho \setminus V} |f(x) - x| > 0$, and define $f_t: U \to \mathbb{R}^k$,

$-a \in B_\varepsilon(0)$ reg pt of $x \mapsto f(x) - x$,

$$f_t(x) = f(x) + t\rho(x) a, \quad x \in U.$$

$\varepsilon = \min(\ (\ ),$ ~~$d(\operatorname{spt}\rho, \partial U)$~~$)$. small enough to carry over to coords.

Then

① $f_t = f$ outside $\operatorname{spt} \rho$.

~~② $f_1 = f + t\rho a$ has only Lefschetz fxd pts.~~

② $f_1 = f + \rho a$ has no fxd pts on $\operatorname{spt}\rho \setminus V$

③ $f_1 = f + a$ on V has only Lef. fxd pts.

Observe that perturbing $\overset{f(z)=}{z + z^m}$ into $\overset{f_1(z)}{z + z^m + a}$, a small, ~~creates m zeroes near 0~~: $f(z) - z = z^m$ has m-multi zero at 0, $f_1(z) - z = z^m + a$ has m simple zeroes near 0. Recall that we were able to detect zeroes by computing the degree of $\frac{f(z) - z}{|f(z) - z|}$.

morning glory

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Re-def. Let $x \in U \overset{\text{open}}{\subseteq} \mathbb{R}^k$ and $f: U \to \mathbb{R}^k$ has an isolated fixed pt ~~at~~ x.

Define the local Lefschetz number of f at x

$$L_x(f) = \deg\left(\frac{f(y) - y}{|f(y) - y|} : \partial B_\varepsilon(x) \to S^{k-1} \right)$$

whenever $\overline{B_\varepsilon(x)} \subset U$ s.t. f has no other fxd pt than x in $\overline{B_\varepsilon(x)}$.

well-def'd : ① indep of choice of ε, boundary thm

② What if f Lefschetz?

$$\deg\left(\frac{df_x(\varepsilon v) - \varepsilon v + o(\varepsilon)}{| \quad \text{''} \quad |} : \partial B_\varepsilon(x) \to S^{k-1} \right)$$

$$= \deg\left(\frac{(df_x - I_k)v}{|(df_x - I_k)v|} : S^{k-1} \to S^{k-1} \right)$$

$$= \operatorname{sgn} \det (df_x - I_k)$$

$\left(\because \right.$ $\det A_0 > 0 \Rightarrow \exists$ htpy A_t, each iso, s.t. $A_1 = I_k$

$\det A_0 < 0 \Rightarrow \exists$ htpy A_t, each iso, s.t. $A_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

or, $GL_k(\mathbb{R})$ has two path-connected components. $\left. \right)$

Prop. If $x \in U \subseteq \mathbb{R}^k$, $f: U \to \mathbb{R}^k$ has iso fxd pt at x.

If f has no other fixed pts in B than x,

and $f_1 : U \to \mathbb{R}^k$ smooth, $f = f_1$ outside B, f_1 has only Lefschetz fixed pts $x_1, \dots, x_N$ in B, then

$$L_x(f) = \sum L_{x_i}(f_1).$$

Def. Let $f: X \to X$ have isolated fixed pt x. Define local Lef number

$$L_x(f) = L_0(\phi^{-1} \circ f \circ \phi)$$

$$\begin{tikzcd} X \arrow[r, "f"] & X \\ U \arrow[u, "\phi"] \arrow[r, dotted] \arrow[ur, phantom, "\circlearrowleft"] & U \arrow[u, "\phi"'] \end{tikzcd}$$

well-def by above prop and splitting prop.

Local Computation of the Lefschetz number.

If $f: X \to X$ has only finitely many fixed pts, then

$$L(f) = \sum_{f(x) = x} L_x(f).$$

Vector Fields and Differential Equations

(Chp 5 of Spivak Vol 1)

Thm. Let $\vec{v}$ be a smooth tangent v.f. on X, and $p \in X$.

Then $\exists\, p \in V^{\text{open}}$, $\varepsilon > 0$ s.t. there is a unique collection of diffeos

$\phi_t : V \to \phi_t(V) \subseteq X$ for $|t| < \varepsilon$ s.t.

(1) $\phi : (-\varepsilon, \varepsilon) \times V \to X$, $\phi(t, q) = \phi_t(q)$ is C^∞

(2) $|s|, |t|, |s+t| < \varepsilon$ and $q, \phi_t(q) \in V$, then

$$\phi_{s+t}(q) = \phi_s(\phi_t(q)).$$

(3) If $q \in V$, ~~then~~ $\cancel{\frac{d}{dt}\Big|_0 \phi_t(q) = \vec{v}(q)}$. $t > 0$, then

~~(this implies~~ $\frac{d}{dt}\phi_t(q) = \vec{v}(\phi_t(q))$ ~~whenever~~ $\cancel{\phi_t(q) \in V}$.

<u>Defn</u> Given vf $\vec{v}$ on X, a curve $\rho : (-\varepsilon, \varepsilon) \to X$ with

$$\rho(0) = p$$
$$\frac{d\rho}{dt} = \vec{v}(\rho(t))$$

is called an integral curve for X w/ initial condition $\rho(0) = p$.

ex/cf. If $X = \mathbb{R}$, $\vec{v}(x) = x^2$, $\rho(0) = 1$, then $\rho(t) = \frac{1}{1-t}$.

So domain of ρ may not be $\mathbb{R}$.

Thm. Let $\vec{v}$ be a smooth vf on <u>cpt</u> X. Then there is a unique collection of diffeos $\phi_t : X \to X$, $t \in \mathbb{R}$, s.t.

(1) $\phi : \mathbb{R} \times X \to X$, $\phi(t, q) = \phi_t(q)$ is C^∞

(2) $\phi_{s+t}(q) = \phi_s(\phi_t(q))$ for $s, t \in \mathbb{R}$, $q \in X$

(3) $\frac{d}{dt}\Big|_0 \phi_t(q) = \vec{v}(q)$ for $q \in X$.

(this implies $\frac{d}{dt}\phi_t(q) = \vec{v}(\phi_t(q))$.

We call ϕ_t a 1-parameter group of diffeomorphisms generated by $\vec{v}$.

Prop. If $f: X \to Y$ diffeo and $\vec{v}$ a vf on X that generates $\{\phi_t\}_{t\in\mathbb{R}}$, then $df(\vec{v})$ is a vf on Y that generates $\{f\circ\phi_t\circ f^{-1}\}_{t\in\mathbb{R}}$.

§3.5 Vector Fields and the Poincaré-Hopf thm.

Def. Let $0 \in U \overset{\text{open}}{\subseteq} \mathbb{R}^k$ and let $\vec{v}: U \to \mathbb{R}^k$ have an isolated zero at 0. Define

$$\operatorname{ind}_0(\vec{v}) = \deg\left(\frac{\vec{v}}{|\vec{v}|} : \partial B_\varepsilon(0) \to S^{k-1}\right)$$ where $\overline{B_\varepsilon} \subseteq U$, only zero of $\vec{v}$ in B_ε is 0.

Well-def by bdry thm. (indep of choice of ε).

Prop. Let $U, \vec{v}, \overline{B_\varepsilon}$ as above. Let $\{f_t : \overline{B_\varepsilon} \to \mathbb{R}^k\}_{0\le t\le t}$ be the flow of $\vec{v}$ s.t.

$\forall\, 0 < t \le t_0$, the only fxd pt of f_t in $\overline{B_\varepsilon}$ is 0.

Then $\operatorname{ind}_0(\vec{v}) = L_0(f_{t_0})$.

pf) Observe that, since $\frac{d}{dt}\big|_0 f_t = \vec{v}$, $f_0 = \mathrm{id}$,

$$f_t(x) = x + t\vec{v}(x) + o(t).$$

So

$$L_0(f_{t_0}) = \deg\left(\frac{f_{t_0}(x) - x}{|f_{t_0}(x) - x|} : \partial B_\varepsilon \to S^{k-1}\right) = \deg\left(\frac{\vec{v}(x) + o(t)}{|\vec{v}(x) + o(t)|} : \partial B_\varepsilon \to S^{k-1}\right)$$

$$= \deg\left(\frac{\vec{v}(x) + o(t^0)}{|\vec{v}(x) + o(t^0)|} : \partial B_\varepsilon \to S^{k-1}\right) = \deg\left(\frac{\vec{v}}{|\vec{v}|} : \partial B_\varepsilon \to S^{k-1}\right) = \operatorname{ind}_0(\vec{v}).$$

<u>Rmk</u> Observe that such a t_0 exists. Indeed,

$\frac{d}{dt} f_t(x) = \vec{v}(f_t(x))$, $\quad \frac{d^2}{dt^2} f_t(x) = \frac{d}{dt}\vec{v}(f_t(x)) = \nabla\vec{v}(f_t(x))\,\vec{v}(f_t(x))$,

$\frac{d}{dt}|\vec{v}(f_t(x))|^2 = 2\,\vec{v}(f_t(x))^t\,\nabla\vec{v}(f_t(x))\,\vec{v}(f_t(x)) \le M|\vec{v}(f_t(x))|^2$.

$\Rightarrow |\vec{v}(f_t(x))|^2 \le e^{tM}|\vec{v}(x)|^2$, so $|\vec{v}(f_t(x))| \le 2|\vec{v}(x)|$ if $t \le \frac{2\log 2}{M}$.

By Taylor, $f_t(x) = x + t\vec{v}(x) + E$, $|E| \le \frac{1}{2}t^2\cdot M|\vec{v}(f_s(x))| \le Mt^2|\vec{v}(x)| \le \frac{t}{2}|\vec{v}(x)|$ if $t \le \frac{1}{2M}$

so that $f_t(x) \ne x$ if $\vec{v}(x) \ne 0$.

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Def. Let $\vec{v}$ be a smooth vf on X w/ an isolated zero at x_0. Define

$$\mathrm{ind}_{x_0}(\vec{v}) = \mathrm{ind}_0(\psi^* v),$$

where $\psi: U \to X$, $0 \mapsto x_0$ is a param., $\psi^* v = (d\psi)^{-1}(v)$

Re-def. Let $\vec{v}$ be a smooth vf on X w/ an isolated zero at x_0.

Let ϕ_t be the flow of $\vec{v}$, let t_0 s.t. $\exists$ ball B around x_0 s.t.

~~$\forall 0<t\le t_0$ x_0 is the only fixed pt of ϕ_t in B.~~

Define $\mathrm{ind}_{x_0}(\vec{v}) = L_{x_0}(\phi_{t_0})$.

well-def: $f_t = \psi^{-1} \circ \phi_t \circ \psi$ is flow of $\psi^* v$, so

$$\mathrm{ind}_0(\psi^* v) = L_0(f_t) = L_{x_0}(\phi_t).$$

Poincaré-Hopf index thm.

If $\vec{v}$ is a smooth vf on a cpt ori mfd X w/ isolated zeroes, then

$$\chi(X) = \sum_{\vec{v}(x)=0} \mathrm{ind}_x(\vec{v}).$$